11. To determine the differential of the logarithmic sine of an arc, or of the function $u = \log \sin x$.
Differentiating, Arts. 9 and 10, we find

$$du = \frac{\cos x\, dx}{R \sin x}, \text{ or, since } \frac{\cos x}{R \sin x} = \frac{1}{\tan x},\ du = \frac{dx}{\tan x}.$$

In a similar manner we shall find the differentials of $u = \log \cos x$, and $u = \log \tan x$, to be, respectively,

$$du = -\frac{dx}{\cot x}, \qquad \text{and } du = \frac{\tan x\, dx}{\sin^2 x}.$$

12. Finally we will take the function $z = \sin^{-1} x$, ... (8) in which z is the arc of which x is the sine. Hence, the equivalent relation will be $x = \sin z$; the differential of which, Art. 10, is

$$dx = \frac{\cos z\, dz}{R}; \text{ whence we have } dz = \frac{R\, dx}{\cos z}.$$

But since $\sin z = x$, and consequently $\cos z = \sqrt{(R^2 - x^2)}$, we have

$$dz = \frac{R\, dx}{\sqrt{(R^2 - x^2)}},$$

which is the differential of (8); i. e., the differential of the arc in functions of its sine. Similarly we shall find the differentials of $z = \cos^{-1} x$, $z = \tan^{-1} x$ &c., to be, respectively,

$$dz = -\frac{R\, dx}{\sqrt{(R^2 - x^2)}}, \qquad dz = \frac{R^2 dx}{R^2 + x^2}, \text{ \&c.}$$

Upon these principles the whole structure of the Differential Calculus may be erected so as to be readily comprehended by the ordinary student.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

Solutions of problems in No. five have been received as follows:

From Marcus Baker, 83, 84 & 85; E. S. Farrow, 84, 85, 86, 87 & 88; J. M. Greenwood and W. H. Baker, 83, 84, 86, 87 & 88; Henry Gunder, 84 & 86; Prof. W. W. Johnson, 86 & 91; Prof. H. T. J. Ludwick, 84, & 86; F. P. Matz, 86; Dr. A. B. Nelson, 84, 86, 87 & 88; O. D. Oathout, 84; Walter Siverly, 83, 84, 86, 87, 88, 89, 90 & 91; E. B. Seitz, 83 84 & 88; Prof. J. Scheffer, 83, 84, 85, 87 & 88; Prof. D. Trowbridge, 86; R. J. Adcock, 89.

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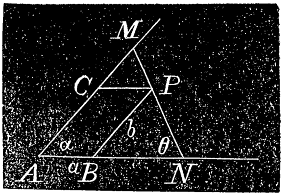
83. "A point is given within two lines which form a given angle with one another. Required the shortest line which can be drawn through this point, terminated by the given lines."

SOLUTION BY M. BAKER, U. S. COAST SURVEY.

From the Fig. $PN = b \sin \alpha \operatorname{cosec} \theta$ and $PM = a \sin \alpha \operatorname{cosec}(\alpha + \theta)$.
$PM + PN = MN = \sin \alpha\,[b \operatorname{cosec} \theta + a \operatorname{cosec}(\alpha + \theta)]$, a minimum.



Differentiating we have $b \cot \theta \operatorname{cosec} \theta + a \cot(\alpha + \theta) \operatorname{cosec}(\alpha + \theta) = 0$;
whence

$$a \cot(\alpha + \theta) = -b \frac{\sin(\alpha + \theta)}{\tan \theta \sin \theta},$$

$$a \frac{1 - \tan \alpha \tan \theta}{\tan \alpha + \tan \theta} = -b \frac{\sin \alpha \cos \theta + \cos \alpha \sin \theta}{\tan \theta \sin \theta} = -b \frac{\sin \alpha + \cos \alpha \tan \theta}{\tan^2 \theta},$$

$a \tan^2 \theta - a \tan \alpha \tan^3 \theta = -b \sin \alpha \tan \alpha - 2b \sin \alpha \tan \theta - b \cos \alpha \tan^2 \theta$,
and finally

$$\tan^3 \theta - \frac{a + b \cos \alpha}{a \tan \alpha} \tan^2 \theta - \frac{2b \sin \alpha}{a \tan \alpha} \tan \theta - \frac{b}{a} \sin \alpha = 0,$$

from which θ may be determined. When $\alpha = 90°$ $\tan \theta = \sqrt[3]{(b \div a)}$.

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84. "The centres of two spheres whose radii are 12 ft. and 5 ft., respectively, are at opposite extremities of the diameter of a circle of 13 ft. radius. Find a point in the circumference of this circle from which the greatest portion of spherical surface is visible.

SOLUTION BY PROF. H. T. J. LUDWICK, SALISBURY, NORTH CAROLINA.

Let x = the distanc of the point from the centre of the larger sphere. Put $a = 12$, $b = 5$, and $d = 13$.

Of the two segments seen, the height of the lager is found $= a - a^2 \div x$, and the height of the smaller $= b - b^2 \div \sqrt{(4d^2 - x^2)}$.

$\therefore$ Surface of segment of larger sphere $= 2\pi a\left(a - \dfrac{a^2}{x}\right)$,

and " " " " smaller " $= 2\pi b\left(b - \dfrac{b^2}{\sqrt{(4d^2 - x^2)}}\right)$.

Hence their sum $S = 2\pi a\left(a - \dfrac{a^2}{x}\right) + 2\pi b\left(b - \dfrac{b^2}{\sqrt{(4d^2 - x^2)}}\right)$.

$$\therefore \quad \frac{dS}{dx} = \frac{a^3}{x^2} + \frac{b^3 x}{\sqrt{[(4d^2 - x^2)^3]}} = 0.$$

$$\therefore \quad x = \frac{2ad}{\sqrt{(a^2 + b^2)}} = \frac{2 . 12 . 13}{\sqrt{(12^2 + 5^2)}} = 24;$$

and $26^2 - 24^2 = 10^2$. Therefor the point is 10 feet from the centre of the smaller sphere.

85. "In a quadrilateral there are given, the length and position of the lower base, the lengths of the two sides, the length of the upper base and the position of a point through which it passes: required to construct the quadrilateral."

[No *construction* of this prob. has been received.—Prof. J. Scheffer writes: "If we describe a circle about one of the extrmities of the lower base of the quadrilateral as a centre, with a radius equal to one of the sides, and another circle about the other extremity of the lower base as a centre, with a radius equal the other side, we reduce the problem to the following:

If two circles are given as to magnitude and position, to lay a line of given length between the two circumferences, which passes throngh a given point.

The line of given length is of course the upper base. As the line may be laid between the outer circumferences, or one outer and one inner circumference, or two inner circumferences, the problem admits in general of four solutions.

This problem belongs to those which cannot be solved by means of the straight line and circle only, and, therefore, exceeds the power of Plane Geometry."

Marcus Baker says that in the solution of 85 the equation is involved to the 8th degree.]

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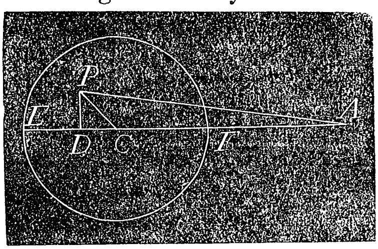
86. "Prove that the attraction of a sphere of uniform density upon an external point is the same as if all the matter of the sphere were concentrated at its centre."

SOLUTION BY A. B. NELSON M. D., DANVILLE, KY.

Let A be the point, and suppose the sphere to be generated by the revolution of a semicircle about its diameter EF. Take the origin of coordinates at the centre C, and let CD and PD be the coordinates of any point P of the semicircle. Then the elemental ring of matter generated by the particle P, $= 2\pi y dy dx$.



Denote CF by 1, and AC by a. The limits of x are $+ 1$ and $- 1$; and of y, 0 and $\surd(1 - x^2)$. The attraction of an elemental ring upon A, measured along CA is

$$\frac{2\pi y dy dx}{AP^2} \times \frac{AD}{AP} = \frac{2\pi y dy dx (a + x)}{[(a+x)^2 + y^2]^{\frac{3}{2}}}.$$

Hence the sum of the attractions of all the particles of the sphere is

$$2\pi\int_{-1}^{+1}(a+x)dx\int_{0}^{\sqrt{(1-x^2)}}\frac{ydy}{[(a+x)^2+y^2]^{\frac{3}{2}}}$$
$$=2\pi\int_{-1}^{+1}(a+x)dx\left(\frac{1}{a+x}-\frac{1}{(a^2+2ax+1)^{\frac{1}{2}}}\right)=\frac{4\pi}{3a^2},$$
which is equal to the attraction of the sphere at the distance a, if all its particles were concentrated at the centre.

[For solution of this problem Mr. Siverly refers to the following works: Simpson's Fluxions p. 450, Emerson's Fluxions p. 370, Vince's Fluxions p. 116, Dealtry's Fluxions p. 146, Poisson's Mechanics Art. 100, Mecanique Celeste Book II, Chap. 2, and Earnshaw's Dynamics p. 333. To which may be added, Newton's Principia Book I, Prop. LXXI.]

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87. "There are n tickets in a bag numbered 1, 2, 3, . . . n. A man draws three tickets together at random and is to receive a number of shillings equal to the product of the numbers he draws. Find the value of his expectation."

SOLUTION BY J. M. GREENWOOD AND W. H. BAKER, KANSAS CITY, MO.

Let s be the sum of the products taken three in a set, p the number of products, and v the value of the chance.

Then $$v=\frac{s}{p} \text{ and } p=\frac{n(n-1)(n-2)}{6} \quad \ldots\ldots\ldots\ldots (1)$$

To find s, (see Todhunter's Algebra, Art. 227), we have

$$(1+2+3+\ldots n)^3=1^3+2^3+3^3+\ldots n^3+3.1^2(2+3+\ldots n)$$
$$+3.2^2(1+3+4+\ldots n)+\ldots\ldots 3.n^2(1+2+3+\ldots n-1)+6s\ldots(2)$$

But $$1+2+3+4+\ldots n=\frac{n(n+1)}{2}, \quad\ldots\ldots\ldots (3)$$

$$1^3+2^3+3^3+4^3+\ldots n^3=\frac{n^2(n+1)^2}{4}, \quad\ldots\ldots\ldots (4)$$

$$3.1^2(2+3+\ldots n)+3.2^2(1+3+4+\ldots n)+\ldots 3.n^2(1+2+3+\ldots n-1)$$
$$=3\,1^2\left(\frac{n(n+1)}{2}-1\right)+3.2^2\left(\frac{n(n+1)}{2}-2\right)+\ldots 3n^2\left(\frac{n(n+1)}{2}-n\right)$$
$$=3.\frac{n(n+1)}{2}.(1^2+2^2+3^2+\ldots n^2)-3(1^3+2^3+3^3+\ldots n^3)$$
$$=\frac{3n^2(n+1)}{2}.\frac{n(n+1)(2n+1)}{6}-\frac{3n^2(n+1)^2}{4}\ldots\ldots\ldots(5)$$

Substituting in (2)

$$\left(\frac{n(n+1)}{2}\right)^3=\frac{n^2(n+1)^2(2n+1)}{4}-\frac{n^2(n+1)^2}{2}+6s.\ \ldots(6)$$

Whence $$s=\frac{n^2(n+1)^2(n-1)(n-2)}{48}; \therefore v=\frac{s}{p}=\frac{n(n+1)^2}{8}.$$

88. "An ellipse revolves about its latus rectum; show that the volumes of the solids generated by the larger and smaller segments are respectively equal to" &c.

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

The equation to the ellipse referred to the latus rectum and major axis, is

$$y = ae \pm \sqrt{\left(a^2 - \frac{x^2}{1-e^2}\right)}.$$

Hence the volume of the solid generated by the larger segment is

$$2\pi\int_0^{a(1-e^2)}\left[ae+\sqrt{\left(a^2-\frac{x^2}{1-e^2}\right)}\right]^2dx + 2\pi\int_{a(1-e^2)}^{a(1-e^2)^{\frac{1}{2}}}\left\{\left[ae\right.\right.$$

$$\left.\left.+\sqrt{\left(a^2-\frac{x^2}{1-e^2}\right)}\right]^2 - \left[ae-\sqrt{\left(a^2-\frac{x^2}{1-e^2}\right)}\right]^2\right\}dx$$

$$= \frac{4\pi a^3}{3}\left(1-e^2\right)\left[\frac{2+e^2}{2} + \frac{3e}{(1-e^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\right];$$

and the volume of the solid generated by the smaller segment is

$$2\pi\int_0^{a(1-e^2)}\left[ae-\sqrt{\left(a^2-\frac{x^2}{1-e^2}\right)}\right]^2dx$$

$$= \frac{4\pi a^3}{3}\left(1-e^2\right)\left[\frac{2+e^2}{2} - \frac{3e}{(1-e^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{1-e}{1+e}\right)^{\frac{1}{2}}\right].$$

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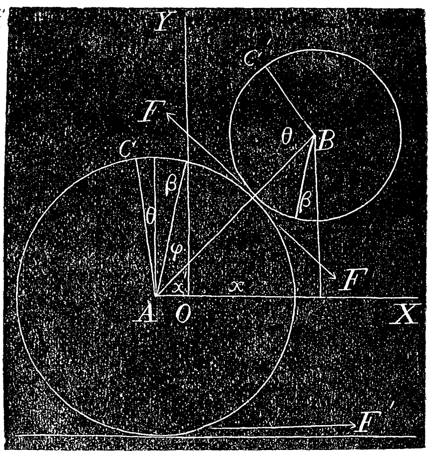
89. "A sphere, radius r, *rolls* down the surface of another sphere of the same material, radius R, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. Determine the motion of the spheres, the point of separation and the equation of the curve described by the center of the upper sphere."

SOLUTION BY WALTER SIVERLY, OIL CITY, PA.



It is evident from the principle of the motion of the center of gravity that the spheres will roll in opposite directions.

Let A and B be the centers of the spheres at any time t after the beginning of the motion, O being the initial position of A; D the point of contact of the spheres; C, C' the points that were in contact at the beginning of the motion; m, m', the masses of A and B respectively; O the origin of horizontal and vertical

coordinates; $(-x, 0)$, (x, y) the coordinates of A and B; F the friction between the spheres, F' the friction between the lower sphere and the plane; θ, θ' the angles through which the spheres have respectively revolved, φ the inclination of AB to the vertical, β being the initial value of φ.

Since F' is the only force acting on the system horizontally,

$$m'\frac{d^2x}{dt^2} - m\frac{d^2x'}{dt^2} = F'. \quad \ldots \quad (1)$$

The rotation of the spheres gives

$$\frac{2}{5}mR\frac{d^2\theta}{dt^2} = F' - F, \quad \ldots \quad (2)$$

$$\frac{2}{5}m'r\frac{d^2\theta}{dt^2} = F. \quad \ldots \quad (3)$$

Since there is perfect rolling,

$$R(\varphi + \theta - \beta) = r(\theta' - \varphi + \beta), \quad \ldots \quad (4)$$

$$x' = R\theta. \quad \ldots \quad (5)$$

Also,

$$x + x' = (R + r)\sin\varphi, \quad \ldots \quad (6)$$

$$y = (R + r)\cos\varphi. \quad \ldots \quad (7)$$

By the principle of *vis viva*,

$$m\left(\frac{2}{5}R^2\frac{d\theta^2}{dt^2} + \frac{dx'^2}{dt^2}\right) + m'\left(\frac{2}{5}r^2\frac{d\theta'^2}{dt^2} + \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right) = 2m'g(c - y)$$

$$= 2m'g(R + r)(\cos\beta - \cos\varphi). \quad \ldots \quad (8)$$

Eliminating F, F' from (1), (2), (3) and integrating once.

$$\frac{2}{5}mR\frac{d\theta}{dt} + \frac{2}{5}m'r\frac{d\theta'}{dt} = m'\frac{dx}{dt} - m\frac{dx'}{dt}. \quad \ldots \quad (9)$$

Eliminating $\frac{dx}{dt}, \frac{dx'}{dt}, \frac{dy}{dt}, \frac{d\theta}{dt}, \frac{d\theta'}{dt}$ from (8), (9), by (4), (5), (6), (7),

$$\left(49m + 45m' - 25m'\cos^2\varphi + 20m'\cos\varphi\right)\frac{d\varphi^2}{dt^2} = \frac{70g}{R + r}\left(m + m'\right)\left(\cos\beta - \cos\varphi\right) \quad \ldots \quad (10)$$

At the point of separation each sphere moves uniformly horizontally, hence

$$\frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0.$$

Differentiating (6) once,

$$\frac{dx}{dt} + \frac{dx'}{dt} = (R + r)\cos\varphi\,.\frac{d\varphi}{dt}.$$

Substituting $\frac{d\varphi}{dt}$ from (10), differentiating and putting $\frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0$, we obtain $25m'\cos^3\varphi - 40m'\cos^2\varphi - [3(49m + 45m') - 20m'\cos\beta]\cos\varphi$

$$= -2\cos\beta\,(49m + 45m'),$$

which determines the point of separation.

Differentiating (6), (7) once and substituting for x' its value from (5),

$$dx + Rd\theta = (R + r)\cos\varphi d\varphi, \quad \ldots\ldots\ldots\ldots (11)$$

$$dy = \quad -(R + r)\sin\varphi d\varphi. \quad \ldots\ldots\ldots\ldots (12)$$

Eliminating θ', x' from (9) by (4), (5), and cancelling dt,

$$(7m + 2m')\,Rd\theta + 2m'(R + r)\,d\varphi = 5m'dx. \quad \ldots\ldots (13)$$

Eliminating $d\theta$ by (11), then $d\varphi$ by (12) and substituting for $\sin\varphi$, $\cos\varphi$, their values $\dfrac{\surd[(R + r)^2 - y^2]}{R + r}$, $\dfrac{y}{R + r}$,

$$7(m + m')\,dx = (-7m + 2m')\,\frac{ydy}{\surd[(R + r)^2 - y^2]} - \frac{2m'(R + r)\,dy}{\surd[(R + r)^2 - y^2]}.$$

Integrating and observing that initially $x = (R+r)\sin\beta$, $y = (R+r)\cos\beta$,

$$7(m+m')x = (7m+2m')\surd[(R+r)^2 - y^2] + 5m'(R+r)\sin\beta + 2m'(R+r)$$
$$\times \{ \cos^{-1}[y \div (R+r)] - \beta \},$$

the equation to the required path.

Mr. Adcock finds for the equation of the curve, (in which α = Siverly's β),

$$x = \frac{2m'(r+R)}{7m+2m'}\left[\sin^{-1}\left(\frac{[2(r+R)y - y^2]^{\frac{1}{2}}}{r+R}\right) - \alpha\right] + \left[2(r+R)y - y^2\right]^{\frac{1}{2}}.$$

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90. "Let an oblate ellipsoid of revolution of homogeneous density rotate about one of its greatest diameters." &c. (See page 160).

SOLUTION BY WALTER SIVERLY.

The attraction of the ellipsoid at any point in the shorter diameter at a distance p from the center, as shown in works on attraction, is

$$4\pi\delta k_1 p\left[\frac{1}{e^2} - \frac{\surd(1 - e^2)}{e^3}\sin^{-1}e\right] = Pp,$$

and the centrifugal force $= p\alpha^2$. Also the attraction at any point on the longer diameter is

$$2\pi\delta k_1 p\left[\frac{\surd(1 - e^2)}{e^3}\sin^{-1}e - \frac{1 - e^2}{e^2}\right] = Qp,$$

and the centrifugal force $= p\alpha^2$. Let u = the pressure at the point distant p from the center on the shorter diameter then

$$du = -(P - \alpha^2)pdp. \qquad u = C - \tfrac{1}{2}p^2(P - \alpha^2).$$

At the surface the pressure = 0.

$\therefore\ C = \frac{1}{2}b^2(P - \alpha^2)$, $u = \frac{1}{2}(b^2 - p^2)(P - \alpha^2)$, $= \frac{1}{2}b^2(P - \alpha^2)$ at the center where $p = 0$. Similarly the pressure at the center on the longer diameter $= \frac{1}{2}a^2(Q - \alpha^2)$. Hence for equilibrium,

$$b^2(P - \alpha^2) = a^2(Q - \alpha^2), \text{ or } (1 - e^2)(P - \alpha^2) = Q - \alpha^2.$$

Substituting values of P and Q and reducing,

$$\frac{3(1-e^2)}{e^2} - \frac{(3-2e^2(_1\ (1-e^2)}{e^3}\sin^{-1}e + \frac{e^2a^2}{2\pi\delta k_1} = 0.$$

When $a^2 \div \delta k_1$ is known the solution of this transcendental equation gives the required ratio of the axes.

[It is well known that there are two ellipsoids of revolution that may be in equilibrium, viz., one of great, and one of small ellipticity. But this question assumes, and the solution is based on the assumption, that an ellipsoid *of revolution* may have its three principal axes unequal. If this is true then there must be an infinite number of ellipsoids of revolution in equilibrium. We have always understood an ellipsoid of revolution, however, to be a spheroid, having all its equatoral diameters equal.—Ed.]

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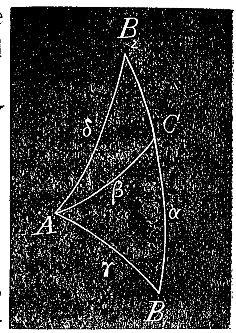
91. "Let a sphere, rotating with the angular velocity w," &c. See p. 88.

SOLUTION BY PROF. W. W. JOHNSON, ANNAPOLIS, MD.

Let A be any particle (the figure being drawn upon the spherical shell in which A is situated) C the original axis, B_1 the new axis, and B_2 the axis of the ring; then $CB_1 = a$, and $B_1B_2 = 90°$. Denote the distances of A from these points as in the fig., and the linear velocities of A, due to the original rotation and the component rotations about B_1 and B_2, by V v_1 and v_2 respectively then



$$V = w\sin\beta,$$
$$v_1 = w\cos a\sin\gamma,$$
$$v_2 = w\sin a\sin\delta.$$

If the linear velocity V is the resultant of v_1 and v_2, there will be no sudden change in the velocity or direction of A. To show that this is the case, it is necessary to prove that the velocities are proportional to the sines of the angles taken respectively each between the directions of the other two. These directions are at right angles to δ, β and γ, and therefore make the same angles which these arcs make with one another. Now denoting the whole angle B_2AB_1 by A, and the part CAB_1 by A_1, we have by spher trigonometry, since $\sin B_1B_2 = 1$,

$$\sin A = \frac{\sin B_1}{\sin\delta}, \quad \text{and} \quad \frac{\sin A_1}{\sin a} = \frac{\sin B_1}{\sin\beta},$$

hence
$$\frac{\sin A}{\sin A_1} = \frac{\sin\beta}{\sin a\sin\delta} = \frac{V}{v_2}.$$

In like manner we may prove $\sin A : \sin A_2 :: V : v_1$.